

UNIVERSITY OF
ILLINOIS LIBRARY
AT URBANA-CHAMPAIGN
BOOKSTACKS



Digitized by the Internet Archive
in 2011 with funding from
University of Illinois Urbana-Champaign

Faculty Working Papers

DYNAMIC DUOPOLY GAME:
DIFFERENTIAL GAME THEORETIC APPROACH

M. Simaan and T. Takayama

#155

College of Commerce and Business Administration
University of Illinois at Urbana-Champaign

FACULTY WORKING PAPERS

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

February 6, 1974

DYNAMIC DUOPOLY GAME:
DIFFERENTIAL GAME THEORETIC APPROACH

M. Simaan and T. Takayama

#155

DYNAMIC DUOPOLY GAME:
DIFFERENTIAL GAME THEORETIC APPROACH[†]

M. Simaan and T. Takayama

ABSTRACT

Output variation type Cournot duopoly game is studied in the framework of differential game theory. Necessary conditions for a general case are developed in their typical coupled form and the difference between an open-loop control (supply adjustment) and a closed-loop control is pointed out. Linear dynamic market demand function and quadratic cost functions case is then studied in detail and analytic expressions for the optimal Cournot controls are obtained in terms of the solution of (1) a set of differential equations for a finite horizon case ($t \in [0, T)$) and (2) a set of algebraic equations for an infinite horizon case ($t \in [0, \infty)$). A numerical example is solved for both optimal Cournot open-loop and closed-loop controls and their respective equilibria. The equilibrium solutions are then compared with the collusive duopolist (optimal control) equilibrium solution.

[†] This work was supported in part by the Joint Services Electronics Program under Contract No. DAAB-07-72-C-0259 and in part by NSF under Grant No. GK-36276 with the Coordinated Science Laboratory, and in part by projects No. 05-382 and 389 with the USDA and the Agricultural Experiment Station at the University of Illinois.

DYNAMIC DUOPOLY GAME:
DIFFERENTIAL GAME THEORETIC APPROACH

M. Simaan and T. Takayama*

Introduction

Static and comparative static formulations of the duopoly game following the line of Cournot (1897) and Stackelberg (1934) flourished in the 1950's after the path-breaking work of von Neumann and Morgenstern (1953), Theory of Games and Economic Behavior. Zero-sum or non-zero-sum, two (or n) person games, cooperative or non-cooperative games, etc. are well documented (see Shubik (1959) for instance). Methodologically, the static and traditional duopoly theory seems to have failed to actively interact with game theoretic approaches (see Osborne (1971)).

It may be worthwhile to point out that logical developments of the traditional duopoly game in its comparative static sense (see Cyert and DeGroot (1973a) and Friedman (1968) for instance) take advantage of a type of sequential decision-making procedures in reaching some reasonable solutions, and have taken a step toward dynamization of the traditional theory. However, they seem to have failed to completely dynamize their models. An interesting venture by Simon, Puig and Aschoff (1973) of

*M. Simaan is Visiting Assistant Professor in the Department of Electrical Engineering and the Coordinated Science Laboratory, and T. Takayama is Professor of Economics, both at the University of Illinois, Urbana-Champaign.

constructing and simulating a dynamic duopoly model led them to a dynamic programming model of the Bellman type and ended with a very negative note that essentially negates all hopes for analytic attempts to answer duopoly questions posed initially by Cournot. We are of the opinion that this negative conclusion is premature and dynamic Cournot duopoly theory is about to emerge as a natural extension of these previous works.

The authors are not aware of the existence of a genuinely dynamic duopoly game model (see Cyert and DeGroot (1973b), and for a "Market Intensity" and two person zero-sum game theoretic approach see Clemhout et al. (1971)). This may be partly due to the fact that any non-cooperative dynamic duopoly game does not seem to fit into the framework of ordinary dynamic optimization models such as optimal control models (see Ho (1970) and Intrilligator (1971)). Thus, this dynamic duopoly problem has to be approached from a different angle, a new angle to be taken up in this paper.

In the first section of this paper, we briefly review static Cournot duopoly games as a lead to the following development of our dynamic duopoly game. In Section 2 we formulate the traditional duopoly game as a typical non-zero sum, non-cooperative differential game problem. Then we formally define two typically supply adjustment decision rules: (i) open-loop control rule, and (ii) closed-loop control rule.

In Section 3 we derive the necessary conditions for the supply adjustment control paths to be optimal for both firms and rules the firms can employ, and discuss their general properties in terms of marginal costs and marginal revenue, etc.

Linear market demand function and quadratic total cost functions case is then analyzed in detail in Section 4, and a solution procedure is developed to generate the optimal open-loop and closed-loop controls for both finite and infinite time horizon problems.

In Section 5, an infinite horizon numerical example is solved and the optimal control paths for the closed-loop and open-loop adjustment rules are obtained, and the differences between these two adjustment rules are pointed out.

In Section 6, collusive duopolistic case, in which both firms act as collusive duopolists, is handled as a typical problem in optimal control theory. The same numerical example treated in Section 5 is then solved for the collusive solution and the three numerical solutions are tabulated for comparison and further investigation.

In the last section, we conclude this paper by pointing out some possible directions of future developments in this interesting area.

1. Static Cournot Duopoly Game

The traditional duopoly game problem can be cast in the following constrained profit maximization framework.

Let $i = 1, 2$ stand for firms 1 and 2, x_i for the output produced and placed in the market by the i th firm, and x for the total marketed quantity, which satisfies the following identity:

$$(1.1) \quad x = x_1 + x_2.$$

It is assumed that the total marketed quantity x and the market price are to satisfy the following well behaved continuous and differentiable (in R_+) functional relationship:

$$(1.2) \quad p = h(x).$$

Total cost functions for both firms are known to each firm as $g_i(x_i)$, $i = 1, 2$. Each firm is then assumed to maximize its profit

$$(1.3) \quad \begin{aligned} \pi_i &= p \cdot x_i - g_i(x_i) \\ &= x_i h(x) - g_i(x_i), \quad i = 1, 2. \end{aligned}$$

The Cournot equilibrium point (x_1^*, x_2^*) is defined by the following pair of inequalities

$$(1.4) \quad \begin{cases} \pi_1(x_1^*, x_2^*) \geq \pi_1(x_1, x_2^*) \\ \pi_2(x_1^*, x_2^*) \geq \pi_2(x_1^*, x_2) \end{cases}$$

and, if typical concavity and differentiability conditions for π_1 and π_2 with respect to x_1 and x_2 respectively are satisfied, it can be obtained by solving the following simultaneous equations:

$$(1.5) \quad \frac{\partial \pi_1}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial \pi_2}{\partial x_2} = 0.$$

The Stackelberg equilibrium point and other equilibrium points such as the Pareto or von Neumann equilibrium point can be readily definable (see Osborne (1971)). However, for the purpose of the present paper the definition above is sufficient, and no further investigation will be carried to other equilibrium concepts at this stage.

2. Dynamic Duopoly Game--Definitions and Formulations

As is customary in most dynamic models, our dynamic duopoly game depends heavily on the system dynamics that works in the market. In this paper, we assume that the market follows the following dynamics:

$$(2.1) \quad \frac{dp(t)}{dt} = \dot{p}(t) = f(p(t), x(t))$$

for $t \in [0, T)$

which states in essence that the rate of change of the market price of the (single) commodity at time t is determined by the present price level, $p(t)$, and the total supply of the commodity by the duopolists,

$$(2.2) \quad x(t) = x_1(t) + x_2(t)$$

where $x_1(t)$ and $x_2(t)$ are outputs or supply quantities by firms 1 and 2 at time t , respectively.

The dynamic demand function (or price adjustment function, the closest terminology we can find in static stability argument of a general equilibrium (see Nikaido (1970))), (2.1), may be justified on the following ground. Consider the situation where $f(p(t), x(t))$ can be written as

$$(2.3) \quad f(p(t), x(t)) = G(h(x(t)) - p(t))$$

where $G(u)$ is a monotone increasing function of the argument and satisfies the following conditions (see Samuelson (1947)):

$$(2.4) \quad G(0) = 0 \quad \text{and} \quad \frac{dG(u)}{du} > 0 \quad \text{for } \forall u \in R.$$

This G function can be considered as the speed of the adjustment function. If, at a certain time $t_1 \in [0, T)$, $h(x(t_1)) - p(t_1) \neq 0$ for some reason such as the market price during its adjustment process was not high (low) enough, resulting in a smaller (larger) supply quantity appearing in the market (see A(B) in Figure 1) than the market actually desired at that price level, then the market price moves in the direction stipulated by (2.1) or (2.3) that is, $\dot{p}(t) > 0$ for A and $\dot{p}(t) < 0$ for B. If we assume that the supply per period remains the same for a while after t_1 , then the price $p(t)$ will move upward (A) (downward (B)) thereafter, along the vertical axis until it reaches $A'(B')$ at time $t_2 > t_1$ where $h(x(t_1)) - p(t_2) = 0$ holds. The rationale above is a genuinely dynamic price adjustment mechanism, and is considered to be a natural extension of the static price response function (see Nikaido (1970)).

In this paper we also assume that

$$(2.5) \quad \begin{cases} (i) & \frac{\partial f}{\partial p} \leq 0 \quad \text{and} \quad \frac{\partial f}{\partial x_i} \leq 0 \text{ for } i = 1, 2, \\ (ii) & f(0, x(t)) \geq 0 \quad \text{for all } x(t) \end{cases}$$

which seem rather acceptable (see Section 5). Condition (2.5 ii) is to insure that the price will never cross the quantity axis to the negative price region.

In the above price adjustment process (or system dynamics) the supply quantity $x(t)$ is treated as an exogenous variable, thus arbitrarily given. However, as is clear from the content of $x(t)$, once the market

price is known, the supply quantity is the sum of the optimum outputs of both firms, $x_1(t)$ and $x_2(t)$, and each optimum supply quantity $x_i(t)$, $i = 1, 2$ is assumed to be the profit maximizing output (trajectory) over time of each firm.

The total profit for the i th firm over the time horizon $[0, T)$ is defined as

$$(2.6) \quad \pi_i(x_1(t), x_2(t)) = \int_0^T e^{-r_i t} [p(t)x_i(t) - g_i(t, x_i(t))] dt \quad \frac{1}{\text{for } i = 1, 2}$$

where

$$(2.7) \quad TC_i \equiv g_i(t, x_i(t)) \quad \text{for } i = 1, 2$$

is the total cost function of the i th firm and r_i is an appropriate discount rate for the i th firm. Thus the optimum supply quantities $x_i(t)$, $i = 1, 2$ over $t \in [0, T)$ can be obtained as the solution of the following dynamic optimization problems

$(2.8) \quad \begin{array}{l} \text{(i) Find } x_1(t) \text{ that maximizes} \\ \pi_1(x_1(t), x_2(t)) \text{ subject to} \\ (2.1) \text{ for } t \in [0, T) \end{array}$	$\begin{array}{l} \text{(ii) Find } x_2(t) \text{ that maximizes} \\ \pi_2(x_1(t), x_2(t)) \text{ subject to} \\ (2.1) \text{ for } t \in [0, T). \end{array}$
--	--

^{1/} In this profit function, the revenues at t , $p(t) x_i(t)$ for $i = 1, 2$ are assumed to be received at the ongoing (disequilibrium) price if the initial price p_0 was in disequilibrium. This might be considered as a loss (gain) in the sales department as a result of accumulation (decumulation) of inventory during the adjustment process when the initial price p_0 was higher (lower) than the equilibrium price for some reason. Introduction of inventory into the system dynamics as another component will be handled in a forthcoming paper.

It is clear from the structure of the twin (more appropriately "coupled") maximization problems above that, it is impossible to solve them separately. This suggests that our problem of solving this dynamic duopoly game is best formulated and studied within the framework of differential game theory (see Kuhn and Szegö (1971), Starr and Ho (1969a), (1969b) and Simaan and Cruz (1973a), (1973b)).

Roughly speaking, a differential game is a system whose states are described by a set of differential equations and where there are more than one decision-maker in the game each trying to optimize a different objective function. In our case, the state of the system is the price $p(t)$, the control variables are the supply quantities $x_1(t)$ and $x_2(t)$, and the objective functions are the profit functions π_1 and π_2 in (2.6).

In this paper we confine our dynamic duopoly model to a homogeneous product and single dynamic market demand function case following the tradition of Cournot. Thus, other models dealing with slightly differentiated product fall outside of the scope of this paper.

It is important at this stage, to point out one of the most interesting features of this dynamic duopoly game. The dynamic time paths of the control variables (supply quantities) $x_1(t)$ and $x_2(t)$ can be constructed in two quite different manners. The first is an open-loop control form that completely ignores the information from the state, $p(t)$, and moves along the prescribed path set at the beginning stage of the game (thus depending on the initial price $p(0)$ and time t only). We write this open-loop path as $(x_{10}(t), x_{20}(t))$.^{2/} The second is a closed-loop

^{2/} Strictly speaking these should be written as $(x_{10}(t, p(0)), x_{20}(t, p(0)))$, but for simplicity and since $p(0)$ is a known constant we will write them as $(x_{10}(t), x_{20}(t))$.

form that takes time path of the state $p(t)$ as additional information (feedback) and adjusts its course as $p(t)$ and t change. The closed-loop control path is written as $(x_{1c}(t, p(t)), x_{2c}(t, p(t)))$. Thus the Cournot behavior of the two firms is defined by the following conditions

$$(2.9) \quad \begin{cases} \pi_1(x_{1o}^*(t), x_{2o}^*(t)) \geq \pi_1(x_{1o}(t), x_{2o}^*(t)) \\ \pi_2(x_{1o}^*(t), x_{2o}^*(t)) \geq \pi_2(x_{1o}^*(t), x_{2o}(t)) \end{cases}$$

for the open-loop path and by

$$(2.10) \quad \begin{cases} \pi_1(x_{1c}^*(t, p(t)), x_{2c}^*(t, p(t))) \geq \pi_1(x_{1c}(t, p(t)), x_{2c}^*(t, p(t))) \\ \pi_2(x_{1c}^*(t, p(t)), x_{2c}^*(t, p(t))) \geq \pi_2(x_{1c}^*(t, p(t)), x_{2c}(t, p(t))) \end{cases}$$

for the closed-loop path. Due to the similarity between (2.9) and (2.10), we call the dynamic equilibrium paths (2.9) and (2.10) as open-loop dynamic Cournot path and closed-loop dynamic Cournot path respectively. Thus the total open-loop market Cournot supply quantity at time t is

$$(2.11) \quad x_o^*(t) = x_{1o}^*(t) + x_{2o}^*(t)$$

and its intersection with the dynamic demand curve $\dot{p}(t) = f(p(t), x_o(t))$ will then yield the optimal open-loop market price $p_o^*(t)$. Similarly the total closed-loop market Cournot supply quantity at time t is

$$(2.12) \quad x_c^*(t, p(t)) = x_{1c}^*(t, p(t)) + x_{2c}^*(t, p(t))$$

and its intersection with the dynamic demand curve $\dot{p}(t) = f(p(t), x_c(t, p(t)))$ will yield the optimal closed-loop market price $p_c^*(t)$.

As will be shown later, and demonstrated by a numerical example, these two different dynamic Cournot paths actually generate two quite different supply responses and their welfare consequences may be argued to be substantially different.

3. Necessary Conditions for Optimality of Dynamic Cournot Paths

In developing necessary conditions for our optimal dynamic Cournot paths, we apply the Pontryagin Maximum Principle (see Pontryagin et al. (1962)).

A. Closed-Loop Controls

Due to the intrinsic virtue of utilizing market price as additional information in guiding its adjustment of the firms' outputs, we develop the conditions for the closed-loop Cournot paths, $x_{ic}^*(t, p(t))$, $i = 1, 2$, first.

Based on the definition of the closed-loop dynamic Cournot paths (2.10) we form the Hamiltonian for the i th firm as follows:

$$(3.1) \quad H_i = p(t)x_i(t, p(t)) - g_i(t, x_i(t, p(t))) + \lambda_i(t)f(p(t), x_1(t, p(t)) + x_2(t, p(t))) \quad , \text{ for } i = 1, 2.$$

Now the necessary conditions (see Arrow (1968)) for the maximizations of (2.11) can be derived as:

$$\begin{aligned}
 & \text{Firm 1} \\
 (3.2) \left\{ \begin{aligned}
 (i) \quad & \dot{p}(t) = f(p(t), x_1(t, p(t)) + x_2(t, p(t))) \quad , \quad p(0) = p_0 \\
 & \dot{\lambda}_1(t) = r\lambda_1(t) - x_1(t, p(t)) - \lambda_1(t) \left(\frac{\partial f}{\partial p} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial p} \right) \\
 & p(t) - \frac{\partial g_1}{\partial x_1} + \lambda_1(t) \frac{\partial f}{\partial x_1} = 0 \\
 & \lambda_1(T) = 0 \quad \underline{3/}
 \end{aligned} \right. \\
 \\
 & \text{Firm 2} \\
 (ii) \left\{ \begin{aligned}
 & \dot{p}(t) = f(p(t), x_1(t, p(t)) + x_2(t, p(t))) \quad , \quad p(0) = p_0 \\
 & \dot{\lambda}_2(t) = r\lambda_2(t) - x_2(t, p(t)) - \lambda_2(t) \left(\frac{\partial f}{\partial p} + \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial p} \right) \\
 & p(t) - \frac{\partial g_2}{\partial x_2} + \lambda_2(t) \frac{\partial f}{\partial x_2} = 0 \\
 & \lambda_2(T) = 0 \quad \underline{3/}
 \end{aligned} \right.
 \end{aligned}$$

Here we assume that $r_1 = r_2 = r$ for simplicity and that g_i , for $i = 1, 2$ and f are such that the Hamiltonians H_1 and H_2 are strictly concave in x_1 and x_2 respectively, so that maximization of H_i by x_i is achieved by setting $\frac{\partial H_i}{\partial x_i} = 0$ for $i = 1, 2$.

Now we face a difficult problem of solving the two point boundary problem consisting of a system of partial differential equations (3.2)

3/ If for a finite horizon problem the profit functions (2.6) were to include a terminal profit term $F_i(p(T))$ accounting for the salvage value of i th firm at the end of the operating period T , then the boundary condition on λ_i in (3.2) should be changed to $\lambda_i(T) = \frac{\partial F_i(p(T))}{\partial p(T)}$, $i = 1, 2$, but for the sake of simplicity, this term is neglected in this paper.

for which in general an analytic solution is extremely difficult to obtain. However in the following two sections, we introduce a linear dynamic demand function and quadratic cost functions and show how to solve analytically the above system of equations for this special case.

In the third equation in (3.2) (i) and (ii), one may be allowed to interpret this relationship as: dynamic marginal revenue $(p(t) + \lambda_i(t) \frac{\partial F}{\partial x_i})$ equals marginal cost $(\frac{\partial g_i}{\partial x_i})$ condition; except at the terminal time T , $\lambda_i(T) = 0$, $i = 1, 2$, must be enforced. A simple conclusion that the terminal marginal costs $\left. \frac{\partial g_i}{\partial x_i} \right|_{x_i(T)}$ must be equal to the terminal price can then be derived.

Under a very mild condition along with (2.5), we can claim that the costate variables $\lambda_i(t)$, $i = 1, 2$, remain nonnegative throughout the time horizon $[0, T)$, (see Appendix 1). This essentially means that the optimal price trajectory $p_c^*(t)$ will always be larger than the marginal costs $\frac{\partial g_i}{\partial x_i}$ (or profits π_i will remain positive along the trajectory).

We now consider the case where the time horizon is infinite. In this case, our interest is also in studying the behavior of the Cournot paths as $t \rightarrow \infty$, and in particular whether limiting values exist or not. As often assumed in infinite horizon problems (see Arrow (1968)), it is appropriate to assume that the cost functions are time invariant, that is

$$(3.3) \quad TC_i = g_i(x_i(t)) \quad , \text{ for } i = 1, 2.$$

The argument behind this assumption according to Arrow is that "the sequence of conditions to be encountered in the future is much the same as today, or can be made so after some simple renormalization." Under

these conditions the Cournot closed-loop supply quantities will also be time invariant of the form $x_{1c}^*(p(t))$, $i = 1, 2$, and the necessary conditions for optimality (see Arrow (1968)) will reduce to:

$$(3.4) \left\{ \begin{array}{l} \text{Firm 1} \\ (i) \left\{ \begin{array}{l} \dot{p}(t) = f(p(t), x_1(p(t)) + x_2(p(t))) \quad , \quad p(0) = p_0 \\ \dot{\lambda}_1(t) = r\lambda_1(t) - x_1(p(t)) + \lambda_1(t) \left(\frac{\partial f}{\partial p} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dp} \right) \\ p(t) - \frac{dg_1}{dx_1} + \lambda_1(t) \frac{\partial f}{\partial x_1} = 0 \\ \lim_{t \rightarrow \infty} e^{-rt} \lambda_1(t) = 0 \end{array} \right. \\ \text{Firm 2} \\ (ii) \left\{ \begin{array}{l} \dot{p}(t) = f(p(t), x_1(p(t)) + x_2(p(t))) \quad , \quad p(0) = p_0 \\ \dot{\lambda}_2(t) = r\lambda_2(t) - x_2(p(t)) + \lambda_2(t) \left(\frac{\partial f}{\partial p} + \frac{\partial f}{\partial x_1} \frac{dx_1}{dp} \right) \\ p(t) - \frac{dg_2}{dx_2} + \lambda_2(t) \frac{\partial f}{\partial x_2} = 0 \\ \lim_{t \rightarrow \infty} e^{-rt} \lambda_2(t) = 0 \end{array} \right. \end{array} \right.$$

Note the change in the terminal conditions for $\lambda_i(t)$, and the use of total derivatives $\frac{dg_i}{dx_i}$ and $\frac{dx_i}{dp}$ instead of their partial derivatives in (3.2).

We now derive very briefly similar conditions for the open-loop controls for finite and infinite horizon problems.

B. Open-Loop Controls

Open-loop dynamic Cournot paths $(x_{10}^*(t), x_{20}^*(t))$ as defined in (2.9) can be derived as follows. First the Hamiltonian of the i th firm is written as:

$$(3.5) \quad H_i = p(t)x_i(t) - g_i(t, x_i(t)) + \gamma_i(t)f(p(t), x_1(t) + x_2(t)), \quad i = 1, 2.$$

and then the necessary conditions for the open-loop Cournot path are given by:

$$(3.6) \quad \left\{ \begin{array}{l} \text{Firm 1} \\ \dot{p}(t) = f(p(t), x_1(t) + x_2(t)) \quad , \quad p(0) = p_0 \\ \dot{\gamma}_1(t) = r\gamma_1(t) - x_1(t) + \gamma_1(t) \frac{\partial f}{\partial p} \\ \text{and} \\ p(t) - \frac{\partial g_1}{\partial x_1} + \gamma_1(t) \frac{\partial f}{\partial x_1} = 0 \\ \gamma_1(T) = 0 \end{array} \right\} \quad \text{for a finite horizon problem}$$

$$\left\{ \begin{array}{l} \text{or} \\ p(t) - \frac{dg_1}{dx_1} + \gamma_1(t) \frac{\partial f}{\partial x_1} = 0 \\ \lim_{t \rightarrow \infty} e^{-rt} \gamma_1(t) = 0 \end{array} \right\} \quad \text{for an infinite horizon problem}$$

$$(3.6) \quad \left\{ \begin{array}{l} \text{Firm 2} \\ \dot{p}(t) = f(p(t), x_1(t) + x_2(t)) \quad , \quad p(0) = p_0 \\ \dot{\gamma}_2(t) = r\gamma_2(t) - x_2(t) + \gamma_2(t) \frac{\partial f}{\partial p} \\ \text{and} \\ p(t) - \frac{\partial g_2}{\partial x_2} + \gamma_2(t) \frac{\partial f}{\partial x_2} = 0 \\ \gamma_2(T) = 0 \end{array} \right\} \quad \text{for a finite horizon problem}$$

$$\left\{ \begin{array}{l} \text{or} \\ p(t) - \frac{dg_2}{dx_2} + \gamma_2(t) \frac{\partial f}{\partial x_2} = 0 \\ \lim_{t \rightarrow \infty} e^{-rt} \gamma_2(t) = 0 \end{array} \right\} \quad \text{for an infinite horizon problem}$$

A distinctive difference between (3.6) and (3.2) or (3.4) is in the second equation: In (3.6), $\frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial p}$ for $i = 1, 2$, is completely missing. This is exactly the term that expresses the awareness (feedback) of the firms of the price level in adjusting their supplies. What would be the effect of this omission cannot be determined in general in analytical terms, although the numerical examples in Sections 5 and 6 show a clear-cut difference and tendency. Also as in the closed-loop case, it can be shown (Appendix 1) that $y_i(t) \geq 0$ for all $t \in [0, T]$.

Solving the coupled differential equations (3.2), (3.4) or (3.6) for general nonlinear $f(p(t), x(t))$ and $g_i(t, x_i(t))$, $i = 1, 2$, is extremely difficult, and in general, one has to resort to numerical techniques using a digital computer. However, this does not suggest that we should abandon analytical attempts for solving and investigating our dynamic Cournot paths.

Thus in the following section we take up a linear dynamic demand function and quadratic total cost functions case and derive an analytic solution for the corresponding dynamic duopoly game problem. This analytic solution will enable us to study some properties of the dynamic Cournot equilibrium that otherwise could not be studied.

4. Linear-Quadratic Dynamic Duopoly Games

In this section, we plan to develop analytical solution procedures to solve for the closed-loop and open-loop dynamic Cournot equilibrium price and supply quantities for both the finite horizon and infinite horizon cases.

The linear dynamic demand function is taken to be of the form

$$(4.1) \quad \dot{p} = c - ap - b(x_1 + x_2) \\ = a\left(\frac{c}{a} - \frac{b}{a}(x_1 + x_2) - p\right) \quad (\text{see (2.3)})$$

where a , b , and c are all positive constants,^{4/} and the quadratic cost functions are assumed to be given as

$$(4.2) \quad g_i = \frac{1}{2} \alpha_i x_i^2 \quad i = 1, 2$$

where $\alpha_i > 0$ for $i = 1, 2$. The overtime profits can then be written as:

$$(4.3) \quad \pi_i = \int_0^T e^{-rt} \left[p x_i - \frac{1}{2} \alpha_i x_i^2 \right] dt, \quad i = 1, 2.$$

Let us first develop the closed-loop solution. After forming the Hamiltonians

$$(4.4) \quad H_i = p(t)x_i(t, p(t)) - \frac{1}{2} \alpha_i x_i^2(t, p(t)) + \lambda_i(t)(c - ap - b(x_1(t, p(t)) \\ + x_2(t, p(t)))) \quad , \quad i = 1, 2$$

we derive the following necessary conditions (3.2) and (3.4):

^{4/} Here we notice that (4.1) satisfies conditions (2.5) only in the region $0 \leq p \leq \frac{c}{a}$ and $0 \leq x_1 + x_2 \leq \frac{c}{b}$. Thus in order to account for these constraints, Kuhn-Tucker conditions can be incorporated in the solution; but since this is not the main issue in this paper, and in order to avoid unnecessary mathematical complications, it will be assumed that the solution obtained will lie in the interior of these constraints, so that the Kuhn-Tucker conditions are no longer needed.

$$\begin{aligned}
 (4.5) \quad & \left\{ \begin{aligned}
 & (i) \quad \dot{p}(t) = c - ap(t) - b(x_1(t, p(t)) + x_2(t, p(t))) \quad , \quad p(0) = p_0 \\
 & (ii) \quad \dot{\lambda}_1(t) = (r + a + b \frac{\partial x_2(t, p(t))}{\partial p}) \lambda_1(t) - x_1(t, p(t)) \\
 & (iii) \quad \dot{\lambda}_2(t) = (r + a + b \frac{\partial x_1(t, p(t))}{\partial p}) \lambda_2(t) - x_2(t, p(t)) \\
 & (iv) \quad p(t) - \alpha_1 x_1(t, p(t)) - b \lambda_1(t) = 0 \\
 & (v) \quad p(t) - \alpha_2 x_2(t, p(t)) - b \lambda_2(t) = 0 \\
 & \text{and} \\
 & (vi) \quad \lambda_1(T) = 0 \quad , \quad \lambda_2(T) = 0 \quad \text{for a finite horizon problem} \\
 & \text{or} \\
 & (vi)' \quad \lim_{t \rightarrow \infty} e^{-rt} \lambda_1(t) = 0 \quad , \quad \lim_{t \rightarrow \infty} e^{-rt} \lambda_2(t) = 0 \\
 & \text{for an infinite horizon problem.}
 \end{aligned} \right.
 \end{aligned}$$

Note that for an infinite horizon problem the supply quantities become time invariant i.e. $x_i(p(t))$ and the partial derivatives in (ii) and (iii) reduce to total derivatives. From (4.5) (i), (iv) and (v), we can deduce that the price movement satisfies

$$(4.6) \quad \dot{p}(t) = [c + \frac{b^2}{\alpha_1} \lambda_1(t) + \frac{b^2}{\alpha_2} \lambda_2(t)] - [a + \frac{b}{\alpha_1} + \frac{b}{\alpha_2}] p(t)$$

or

$$\dot{p}(t) = \theta(t) - \beta p(t)$$

where $\beta > 0$ and constant. The solution of (4.6) is

$$p_c^*(t) = e^{-\beta t} p_0 + \int_0^t e^{-\beta(t-\tau)} \theta(\tau) d\tau.$$

and therefore it can be easily seen that $p_c^*(t)$ depends not on $\lambda_1(t)$ and $\lambda_2(t)$ but on their time histories over the interval $[0, t)$. Furthermore

if $\lambda_1(t)$ and $\lambda_2(t)$ converge within the time horizon to positive constants $\bar{\lambda}_1$ and $\bar{\lambda}_2$, then we can easily see that $p_c^*(t)$ converges to

$$(4.8) \quad \bar{p}_c = \frac{c + \left(\frac{\bar{\lambda}_1}{\alpha_1} + \frac{\bar{\lambda}_2}{\alpha_2}\right)b^2}{a + \frac{b}{\alpha_1} + \frac{b}{\alpha_2}} > 0.$$

Now we focus our attention on $\lambda_1(t)$ and $\lambda_2(t)$ to find if there is a functional relationship between them and $p(t)$ that will simplify the solution of (4.5) and possibly reduce it from a set of partial differential equations to a set of ordinary differential equations. This is achieved by the form:

$$(4.9) \quad \lambda_i(t) = K_i(t)p(t) + E_i(t) \quad \underline{5}/ \quad i = 1, 2$$

where $K_i(t)$ and $E_i(t)$ are functions of time to be determined. Once this relationship is established, the closed-loop controls are obtained from (4.5) (iv) and (v) as:

$$(4.10) \quad x_{ic}^*(t, p(t)) = \frac{1}{\alpha_i} (1 - bK_i)p(t) - \frac{b}{\alpha_i} E_i, \quad i = 1, 2.$$

In order to determine K_i and E_i , $i = 1, 2$, we differentiate (4.9) to get:

$$(4.11) \quad \dot{\lambda}_i = \dot{K}_i p + K_i \dot{p} + \dot{E}_i$$

and from (4.5) (i) and (4.10) we get

^{5/} For simplicity, we will drop the time dependence expression t from λ_i , K_i and E_i , $i = 1, 2$; hereafter unless stated otherwise.

$$\begin{aligned}
 (4.12) \quad \dot{p} &= (c + \frac{b^2}{\alpha_1^2} E_1 + \frac{b^2}{\alpha_2^2} E_2) - (a + \frac{b}{\alpha_1} (1 - bK_1) + \frac{b}{\alpha_2} (1 - bK_2))p(t) \\
 &= p_c(t) - \sigma_c(t)p(t). \quad \text{6/}
 \end{aligned}$$

Now using (4.5) (ii) we get

$$\begin{aligned}
 \dot{\lambda}_1 &= \dot{K}_1 p + K_1 \dot{p}_c - K_1 \sigma_c p + \dot{E}_1 \\
 &= (r + a + \frac{b}{\alpha_2} (1 - bK_2))(K_1 p + E_1) - \frac{1}{\alpha_1} (1 - bK_1)p + \frac{b}{\alpha_1} E_1.
 \end{aligned}$$

Grouping terms we get

$$\begin{aligned}
 &[\dot{K}_1 - K_1 \sigma_c - (r + a + \frac{b}{\alpha_2} (1 - bK_1))K_1 + \frac{1}{\alpha_1} (1 - bK_1)]p \\
 &+ [K_1 \dot{p}_c + \dot{E}_1 - (r + a + \frac{b}{\alpha_2} (1 - bK_2))E_1 - \frac{b}{\alpha_1} E_1] = 0
 \end{aligned}$$

which must hold for all p , implying that

$$(4.13) \quad (i) \quad \begin{cases} \dot{K}_1 = K_1(a + \frac{b}{\alpha_1}(1 - bK_1) + \frac{b}{\alpha_2}(1 - bK_2)) + (r + a + \frac{b}{\alpha_2}(1 - bK_2))K_1 - \frac{1}{\alpha_1}(1 - bK_1) \\ \dot{E}_1 = (r + a + \frac{b}{\alpha_2}(1 - bK_2))E_1 + \frac{b}{\alpha_1} E_1 - K_1(c + \frac{b^2}{\alpha_1^2} E_1 + \frac{b^2}{\alpha_2^2} E_2). \end{cases}$$

Similarly from (4.5) (iii) we get

$$(4.13) \quad (ii) \quad \begin{cases} \dot{K}_2 = K_2(a + \frac{b}{\alpha_1}(1 - bK_1) + \frac{b}{\alpha_2}(1 - bK_2)) + (r + a + \frac{b}{\alpha_1}(1 - bK_1))K_2 - \frac{1}{\alpha_2}(1 - bK_2) \\ \dot{E}_2 = (r + a + \frac{b}{\alpha_1}(1 - bK_1))E_2 + \frac{b}{\alpha_2} E_2 - K_2(c + \frac{b^2}{\alpha_1^2} E_1 + \frac{b^2}{\alpha_2^2} E_2). \end{cases}$$

6/ It will be shown later that for an infinite horizon problem, $\sigma_c(t)$ and $p_c(t)$ become constants $\bar{\sigma}_c$ and $\bar{p}_c$ and the price $p_c^*(t)$ will asymptotically converge to $\bar{p}_c = \bar{p}_c / \bar{\sigma}_c$.

We also note that $\lambda_1(T) = \lambda_2(T) = 0$ imply that $K_1(T) = K_2(T) = E_1(T) = E_2(T) = 0$. These will serve as boundary conditions for the above differential equations. Thus K_i and E_i in (4.10) are obtained by solving backward in time the system of coupled ordinary differential equations (4.13). It is interesting to note that the equations for K_1 and K_2 are not coupled with those of E_1 and E_2 , and therefore one can solve for K_1 and K_2 first, and then for E_1 and E_2 . Furthermore the equations for K_1 and K_2 are of the Riccati type (quadratic) and numerical programs for their solution on a digital computer are readily available. The equations for E_1 and E_2 are linear, and are easy to solve.

When the time horizon is infinite, it can be shown that $K_i(t)$ and $E_i(t)$ will become constants ($\dot{K}_i = \dot{E}_i = 0$) and the dynamic Cournot supply quantities are given by

$$(4.14) \quad x_{ic}^*(p(t)) = \frac{1}{\alpha_i} (1 - b\bar{K}_i) p(t) - \frac{b}{\alpha_i} \bar{E}_i, \quad i = 1, 2$$

where $\bar{K}_1$, $\bar{K}_2$, $\bar{E}_1$ and $\bar{E}_2$ are solutions of the following algebraic equations obtained from (4.13) by setting $\dot{K}_i = \dot{E}_i = 0$.

$$(4.15) \quad \left\{ \begin{array}{ll} (a) & K_1 \left[r + 2a + \frac{b}{\alpha_1} (1 - bK_1) + 2 \frac{b}{\alpha_2} (1 - bK_2) \right] - \frac{1}{\alpha_1} (1 - bK_1) = 0 \\ (a') & K_2 \left[r + 2a + \frac{b}{\alpha_2} (1 - bK_2) + 2 \frac{b}{\alpha_1} (1 - bK_1) \right] - \frac{1}{\alpha_2} (1 - bK_2) = 0 \\ (b) & \left[r + a + \frac{b}{\alpha_1} (1 - bK_1) + \frac{b}{\alpha_2} (1 - bK_2) \right] E_1 - K_1 \frac{b^2}{\alpha_2} E_2 - K_1 c = 0 \\ (b') & \left[r + a + \frac{b}{\alpha_1} (1 - bK_1) + \frac{b}{\alpha_2} (1 - bK_2) \right] E_2 - K_2 \frac{b^2}{\alpha_1} E_1 - K_2 c = 0 \end{array} \right.$$

and (4.12) becomes

$$\begin{aligned}
 (4.16) \quad \dot{p} &= (c + \frac{b^2}{\alpha_1} \bar{E}_1 + \frac{b^2}{\alpha_2} \bar{E}_2) - (a + \frac{b}{\alpha_1} (1 - b\bar{K}_1) + \frac{b}{\alpha_2} (1 - b\bar{K}_2))p \\
 &= \bar{p}_c - \bar{\sigma}_c p .
 \end{aligned}$$

There are two questions, however, that still need to be answered:

- i) Does the system of equations (4.15) have a solution?
- ii) Are the boundary conditions (4.5) (vi)' on $\lambda_i(t)$ satisfied?

In order to answer these questions, we need the following proposition:

Proposition 1: Equations (4.15) have a solution $\bar{K}_1$, $\bar{K}_2$, $\bar{E}_1$, and $\bar{E}_2$, that satisfies

$$0 \leq \bar{K}_1 \leq \frac{1}{b} \quad , \quad 0 \leq \bar{K}_2 \leq \frac{1}{b} .$$

Proof: The proof is given in Appendix 2.

Thus, the above proposition guarantees the existence of a solution to (4.15) which has the property that $\bar{\sigma}_c$ in (4.16) is positive. Therefore as $t \rightarrow \infty$, the price $p_c^*(t)$ (which is the solution of (4.16)) will approach asymptotically a constant value

$$\bar{p}_c = \frac{\bar{p}_c}{\bar{\sigma}_c}$$

and the boundary conditions (4.5) (vi)' are automatically satisfied

$$(4.17) \quad \lim_{t \rightarrow \infty} e^{-rt} \lambda_i(t) = \lim_{t \rightarrow \infty} e^{-rt} (\bar{K}_i p_c^*(t) + \bar{E}_i) = 0.$$

One of the common features of the above finite and infinite horizon solutions is that in both cases the supply quantities are linear functions

of the price, however, in the finite horizon case these linear functions are time varying (4.10) while in the infinite horizon case (4.14) they are not.

We stated at the beginning of this paper that the closed-loop and open-loop supply curves are generally quite different. In the following we plan to develop the open-loop solution for the finite and infinite horizon cases of the linear quadratic dynamic duopoly game given by (4.1)-(4.3). The Hamiltonians of the system are:

$$(4.18) \quad \begin{cases} H_1 = p(t)x_1(t) - \frac{1}{2} \alpha_1 x_1^2(t) + \gamma_1(t)(c - ap(t) - b(x_1(t) + x_2(t))) \\ H_2 = p(t)x_2(t) - \frac{1}{2} \alpha_2 x_2^2(t) + \gamma_2(t)(c - ap(t) - b(x_1(t) + x_2(t))). \end{cases}$$

Note that the supply quantities x_1 and x_2 are no longer functions of $p(t)$. Following (3.6), the necessary conditions for the open-loop solutions are:

$$(4.19) \quad \left\{ \begin{array}{ll} \text{(i)} & \dot{p}(t) = c - ap(t) - b(x_1(t) + x_2(t)) \quad , \quad p(0) = p_0 \\ \text{(ii)} & \dot{\gamma}_1(t) = (r+a)\gamma_1(t) - x_1(t) \\ \text{(iii)} & \dot{\gamma}_2(t) = (r+a)\gamma_2(t) - x_2(t) \\ \text{(iv)} & p(t) - \alpha_1 x_1(t) - b\gamma_1(t) = 0 \\ \text{(v)} & p(t) - \alpha_2 x_2(t) - b\gamma_2(t) = 0 \\ \text{(vi)} & \gamma_1(T) = 0 \quad \text{and} \quad \gamma_2(T) = 0 \quad \text{for a finite horizon problem} \\ \text{or} & \\ \text{(vi)'} & \lim_{t \rightarrow \infty} e^{-rt} \gamma_1(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} e^{-rt} \gamma_2(t) = 0 \\ & \text{for an infinite horizon problem.} \end{array} \right.$$

As in the closed-loop case, it is easy to see that the open-loop price movement will follow the differential equation

$$\dot{p} = (c + \frac{b^2}{\alpha_1} \gamma_1(t) + \frac{b^2}{\alpha_2} \gamma_2(t)) - (a + \frac{b}{\alpha_1} + \frac{b}{\alpha_2}) p(t),$$

and that if $\gamma_1(t)$ and $\gamma_2(t)$ converge within the time horizon to $\bar{\gamma}_1$ and $\bar{\gamma}_2$, positive constants; then the open-loop price will converge to

$$(4.20) \quad \bar{p}_0 = \frac{c + (\frac{\bar{\gamma}_1}{\alpha_1} + \frac{\bar{\gamma}_2}{\alpha_2})b^2}{a + \frac{b}{\alpha_1} + \frac{b}{\alpha_2}} > 0.$$

Whether we have an infinite or a finite horizon problem, Eq. (4.19) can be solved directly, as a two point boundary value problem, for $x_{10}^*(t)$ and $x_{20}^*(t)$. Naturally these supply functions will depend implicitly on p_0 ; and for each value of p_0 , (4.19) has to be solved repeatedly for the corresponding open-loop supply functions. This numerically cumbersome procedure, however, can be avoided by noting, as in the closed-loop case, that $\gamma_i(t)$ can be expressed in terms of $p(t)$ by an expression of the form

$$(4.21) \quad \gamma_i(t) = D_i(t)p(t) + F_i(t), \quad i = 1, 2.$$

The price at time t then will satisfy the differential equation

$$(4.22) \quad \begin{aligned} \dot{p} &= (c + \frac{b^2}{\alpha_1} F_1 + \frac{b^2}{\alpha_2} F_2) - (a + \frac{b}{\alpha_1} (1 - bD_1) + \frac{b}{\alpha_2} (1 - bD_2)) p(t) \\ &= p_0(t) - \sigma_0(t)p(t) \end{aligned}$$

and its solutions gives the open-loop Cournot price:

$$(4.23) \quad p_o^*(t) = e^{-\int_0^t \sigma_o(\tau) d\tau} p_o + \int_0^t e^{-\int_\tau^t \sigma_o(\lambda) d\lambda} \rho_o(\tau) d\tau.$$

The open-loop supply functions can then be computed from (4.19) (iv) and (v) as

$$(4.24) \quad x_{i0}^*(t) = \frac{1}{\alpha_i} (p_o^*(t) - b\gamma_i(t))$$

$$= \frac{1}{\alpha_i} (1-bD_i) \left(e^{-\int_0^t \sigma_o(\tau) d\tau} p_o + \int_0^t e^{-\int_\tau^t \sigma_o(\lambda) d\lambda} \rho_o(\tau) d\tau \right) - \frac{b}{\alpha_i} F_i$$

$$i = 1, 2.$$

Now in order to determine $D_i(t)$ and $F_i(t)$, we follow a procedure similar to that used for determining $K_i(t)$ and $E_i(t)$ of the closed-loop solution. Then it can be easily shown that $D_i(t)$ and $F_i(t)$, $i = 1, 2$ must satisfy the differential equations

$$(4.25) \quad \left\{ \begin{array}{l} (a) \quad \dot{D}_1 = D_1(r+2a + \frac{b}{\alpha_1}(1-bD_1) + \frac{b}{\alpha_2}(1-bD_2)) - \frac{1}{\alpha_1}(1-bD_1) \\ (a)' \quad \dot{D}_2 = D_2(r+2a + \frac{b}{\alpha_1}(1-bD_1) + \frac{b}{\alpha_2}(1-bD_2)) - \frac{1}{\alpha_2}(1-bD_2) \\ (b) \quad \dot{F}_1 = (r+a + \frac{b}{\alpha_1}(1-bD_1))F_1 - \frac{b^2}{\alpha_2} D_1 F_2 - D_1 c \\ (b)' \quad \dot{F}_2 = (r+a + \frac{b}{\alpha_2}(1-bD_2))F_2 - \frac{b^2}{\alpha_1} D_2 F_1 - D_2 c \end{array} \right.$$

with the boundary conditions $D_1(T) = D_2(T) = F_1(T) = F_2(T) = 0$. The equations for D_1 and D_2 are again recognized to be of the Riccati type.

When the time horizon is infinite, it can be easily shown that (4.19) will be satisfied if the supply functions are of the form

$$\begin{aligned}
 (4.26) \quad x_{i0}^*(t) &= \frac{1}{\alpha_i} (1-b\bar{D}_i) \left\{ e^{-\bar{\sigma}_o t} p_o + \int_0^t e^{-\bar{\sigma}_o (t-\tau)} \bar{p}_o d\tau \right\} - \frac{b}{\alpha_i} \bar{F}_i \\
 &= \frac{1}{\alpha_i} (1-b\bar{D}_i) \left\{ e^{-\bar{\sigma}_o t} \left(p_o - \frac{\bar{p}_o}{\bar{\sigma}_o} \right) + \frac{\bar{p}_o}{\bar{\sigma}_o} \right\} - \frac{b}{\alpha_i} \bar{F}_i \\
 &= \frac{1}{\alpha_i} (1-b\bar{D}_i) \left(p_o - \frac{\bar{p}_o}{\bar{\sigma}_o} \right) e^{-\bar{\sigma}_o t} + \frac{1}{\alpha_i} (1-b\bar{D}_i) \frac{\bar{p}_o}{\bar{\sigma}_o} - \frac{b}{\alpha_i} \bar{F}_i
 \end{aligned}$$

where (4.22):

$$\bar{p}_o = c + \frac{b^2}{\alpha_1} \bar{F}_1 + \frac{b^2}{\alpha_2} \bar{F}_2$$

and

$$\bar{\sigma}_o = a + \frac{b}{\alpha_1} (1-b\bar{D}_1) + \frac{b}{\alpha_2} (1-b\bar{D}_2),$$

and where $\bar{D}_1$, $\bar{D}_2$, $\bar{F}_1$ and $\bar{F}_2$ are a solution of the following algebraic set of equations obtained from (4.25) by setting $\dot{D}_i = \dot{F}_i = 0$:

$$(4.27) \quad \left\{ \begin{aligned}
 &D_1(r+2a + \frac{b}{\alpha_1} (1-bD_1) + \frac{b}{\alpha_2} (1-bD_2)) - \frac{1}{\alpha_1} (1-bD_1) = 0 \\
 &D_2(r+2a + \frac{b}{\alpha_1} (1-bD_1) + \frac{b}{\alpha_2} (1-bD_2)) - \frac{1}{\alpha_2} (1-bD_2) = 0 \\
 &(r+a + \frac{b}{\alpha_1} (1-bD_1))F_1 - \frac{b^2}{\alpha_2} D_1F_2 - D_1c = 0 \\
 &(r+a + \frac{b}{\alpha_2} (1-bD_2))F_2 - \frac{b^2}{\alpha_1} D_2F_1 - D_2c = 0.
 \end{aligned} \right.$$

As in the closed-loop case, there are also two questions that need to be answered for this infinite horizon solution:

i) does the system of equations (4.27) have a solution?

ii) are the boundary conditions (4.19) (vi)' satisfied?

In order to answer these questions, we need the following proposition.

Proposition 2: Equations (4.27) have a solution $\bar{D}_1$, $\bar{D}_2$, $\bar{F}_1$ and $\bar{F}_2$ that satisfies

$$0 \leq \bar{D}_1 \leq \frac{1}{b}, \quad 0 \leq \bar{D}_2 \leq \frac{1}{b}.$$

Proof: The proof is given in Appendix 2.

Thus the above proposition guarantees the existence of a solution to (4.27) which has the property that $\bar{\sigma}_0$ is positive and constant. Hence as $t \rightarrow \infty$, the price $p_0^*(t)$ (4.22 - 4.23) will approach asymptotically $\bar{p}_0 = \bar{p}_0 / \bar{\sigma}_0$ and the boundary conditions (4.19) (vi)' will be automatically satisfied

$$\lim_{t \rightarrow \infty} e^{-rt} \gamma_i(t) = \lim_{t \rightarrow \infty} e^{-rt} (\bar{D}_i p_0^*(t) + \bar{F}_i) = 0.$$

Comparing the above closed-loop and open-loop solutions it is now clear that $p_c^*(t)$ is different from $p_o^*(t)$. The effect of the missing term $b \frac{\partial x_i}{\partial p}$, $i = 1, 2$, in (4.19) (ii) and (iii) is still difficult to trace out analytically. However, unless $b = 0$, which is a trivial and almost meaningless case in our dynamic duopoly game, this effect must appear somehow as will be clearly seen in the numerical examples in Section 5 to follow.

5. A Numerical Example

In this section we show how the infinite time horizon closed-loop and open-loop solutions are obtained for our dynamic Cournot duopoly game with the following dynamic demand function:

$$(5.1) \quad \dot{p} = 4 - 2p - (x_1 + x_2)$$

and the following cost functions

$$(5.2) \quad g_1 = \frac{1}{3} x_1^2 \quad \text{and} \quad g_2 = \frac{1}{2} x_2^2.$$

For a 10% discount rate, the infinite horizon profits are:

$$(5.3) \quad \begin{cases} \pi_1 = \int_0^{\infty} e^{-0.1t} [px_1 - \frac{1}{3} x_1^2] dt \\ \pi_2 = \int_0^{\infty} e^{-0.1t} [px_2 - \frac{1}{2} x_2^2] dt. \end{cases}$$

Comparing these expressions with (4.1)-(4.3) we see that the parameter values are: $a = 2$, $b = 1$, $c = 4$, $\alpha_1 = 2/3$, $\alpha_2 = 1$ and $r = 0.1$. Thus the equations for K_1 and K_2 in (4.15)(a), (a)' in the closed-loop solution become

$$(5.4) \quad \begin{cases} (a) \quad 1.5K_1^2 + 2K_1K_2 - 9.1K_2 + 1.5 = 0 \\ (a)' \quad K_2^2 + 3K_1K_2 - 9.1K_2 + 1 = 0 \end{cases}$$

which are solved simultaneously for

$$(5.5) \quad \bar{K}_1 = 0.17 \quad \text{and} \quad \bar{K}_2 = 0.12.$$

This pair clearly satisfies the conditions of Proposition 1. When these values are substituted in (4.15) (b)-(b)' the following equations are obtained

$$(5.6) \quad \begin{cases} (b) & 4.22E_1 - 0.17E_2 = 0.68 \\ (b)' & -0.18E_1 + 4.22E_2 = 0.48 \end{cases}$$

and their solutions are

$$(5.7) \quad \bar{E}_1 = 0.165 \quad \text{and} \quad \bar{E}_2 = 0.126.$$

Now by (4.14), we get the closed-loop Cournot supply functions for firms 1 and 2 as follows:

$$(5.8) \quad \begin{cases} x_{1c}^*(p(t)) = -0.248 + 1.25 p(t) \\ x_{2c}^*(p(t)) = -0.126 + 0.88 p(t). \end{cases}$$

The price change, $\dot{p}$, can now be written by using (5.8) and (4.12) as a function of p only:

$$(5.9) \quad \dot{p}(t) = 4.374 - 4.13 p(t).$$

This gives us the equilibrium price $\bar{p}_c$ when $t \rightarrow \infty$:

$$(5.10) \quad \bar{p}_c = \frac{4.374}{4.13} = 1.06,$$

and the time path of the price when the initial price is given as p_0 is

$$(5.11) \quad p_C^*(t) = (p_0 - 1.06)e^{-\frac{1}{4} 13t} + 1.06.$$

Figure 2 shows the time paths of $p_C^*(t)$ when $p_0 = p_0^1 = 2$ and $p_0 = p_0^2 = 1/2$. Also we can draw a static duopoly game analogue of quantity-price relationships by using (5.1) and (5.8) as shown in Figure 3. Thus as is clear from the above development, when the initial price, say, $p_0^1 = 2$, the optimal supply rate is shown as $x_C^*(p_0^1)$. The price change is obviously negative and as a result the price will decrease. However as this is happening, the total supply will also be adjusted by (5.8) and hence both price and total supply will move in the negative direction along $\overline{x_C^*(p_0^1)}, C$ until the closed-loop equilibrium point C is reached. Similarly, if the initial price is $p_0^2 = 0.5$ then the price change will be upward along $\overline{x_C^*(p_0^2)}, C$ until point C is reached.

Let us now solve the numerical example given above for the open-loop Cournot controls. Substituting the numerical values of the given parameters in (4.25) and solving the resulting equations we get

$$\begin{aligned} \bar{D}_1 &= 0.19 & , & \bar{D}_2 = 0.14 \\ \bar{F}_1 &= 0.251 & , & \bar{F}_2 = 0.207. \end{aligned}$$

The dynamic demand function (4.22) reduces to

$$(5.12) \quad \dot{p}(t) = 4.584 - 4.07 p(t)$$

and the equilibrium open-loop price as $t \rightarrow \infty$ is

$$\bar{p}_0 = \frac{4.584}{4.07} = 1.13.$$

The time path of the open-loop price when the initial price is p_0 is

$$(5.13) \quad p_0^*(t) = (p_0 - 1.13)e^{-4.07t} + 1.13,$$

and the open-loop Cournot supply quantities are (4.24):

$$(5.14) \quad \begin{cases} x_{10}^*(t) = 1.21(p_0 - 1.13)e^{-4.07t} + 0.98 \\ x_{20}^*(t) = 0.86(p_0 - 1.13)e^{-4.07t} + 0.76 . \end{cases}$$

Welfare implications of these two results may be of some significance if any generalization such as to follow is allowed. The closed-loop control law keeps both firms in the market aware of the reaction of the price due to their output decisions and allows them to adjust their outputs based on the current price in the market. The open-loop controls, on the other hand, provide for an initial programming of the outputs of both firms over the time horizon and does not allow for any deviations from these values. Certainly if the price can be measured at each instant t then the closed-loop controls are more desirable than the open-loop controls. Furthermore from the above numerical example, one may be able to conclude that, given a sufficiently long adjustment period, (1) the closed-loop equilibrium price is lower than its counterpart, the open-loop equilibrium price, and (2) the closed-loop equilibrium quantity supplied is also larger than its open-loop counterpart. Thus, the consumer's welfare may, in general, be higher in the closed-loop equilibrium than in the open-loop equilibrium.

6. Optimal Supply Control of Collusive Duopolists

When the two firms take a collusive action in adjusting their supply in the way to maximize their joint profit, our differential game collapses to the following optimal control problem:

Find the optimal supply functions $(x_1^*(t), x_2^*(t))$ that maximize

$$(6.1) \quad \begin{aligned} \pi_c &= \pi_1 + \pi_2 \\ &= \int_0^T e^{-rt} [p(t)(x_1(t) + x_2(t)) - g_1(t, x_1(t)) - g_2(t, x_2(t))] dt \end{aligned}$$

subject to

$$(6.2) \quad \dot{p}(t) = f(p(t), x_1(t) + x_2(t)) \quad , \quad p(0) = p_0.$$

By forming the Hamiltonian

$$(6.3) \quad \begin{aligned} H_c &= p(t)(x_1(t) + x_2(t)) - g_1(t, x_1(t)) - g_2(t, x_2(t)) \\ &\quad + \xi(t)f(p(t), x_1(t) + x_2(t)) \end{aligned}$$

we can easily derive the following necessary conditions for optimality ^{6/}

$$(6.4) \quad \left\{ \begin{array}{ll} \text{(i)} & \dot{p} = f(p, x_1 + x_2) \quad , \quad p(0) = p_0 \\ \text{(ii)} & \dot{\xi} = (r + \frac{\partial f}{\partial p})\xi - (x_1 + x_2) \\ \text{(iii)} & p - \frac{\partial g_1}{\partial x_1} + \xi \frac{\partial f}{\partial x_1} = 0 \\ \text{(iv)} & p - \frac{\partial g_2}{\partial x_2} + \xi \frac{\partial f}{\partial x_2} = 0 \end{array} \right.$$

^{6/} Obviously, since this is an optimal control problem, the open-loop and closed-loop solutions lead to the same optimal paths. For obvious reasons however, we will derive the control in closed-loop form.

$$(6.4) \quad \left\{ \begin{array}{ll} \text{and} & \\ (v) \quad \xi(T) = 0 & \text{for a finite horizon problem} \\ \text{or} & \\ (v)' \quad \lim_{t \rightarrow \infty} e^{-rt} \xi(t) = 0 & \text{for an infinite horizon problem.} \end{array} \right.$$

Since $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2}$, the third and fourth relations yield an interesting result in that

$$(6.5) \quad \frac{\partial g_1}{\partial x_1} = \frac{\partial g_2}{\partial x_2},$$

that is, in order to maximize their joint profit, the two firms must adjust their supplies in such a way that along the optimal path, their marginal costs are equal. Furthermore it can be easily proved that $\xi(t) \geq 0$ for all $t \in [0, T]$ (see Appendix 1) which means that the price will always be larger than the marginal cost of either firm, and that $p(T) = \frac{\partial g_1}{\partial x_1} \Big|_{t=T} = \frac{\partial g_2}{\partial x_2} \Big|_{t=T}$.

Applying these conditions to the linear-quadratic case (4.1)-(4.2), it follows from (6.5) that

$$(6.6) \quad \alpha_1 x_1^*(t) = \alpha_2 x_2^*(t)$$

which means that the optimum output of the first firm is exactly (α_2/α_1) of that of the second firm. Thus, the more efficient firm gets the larger (fixed) market share. The necessary conditions (6.4) reduce to:

$$(6.7) \left\{ \begin{array}{ll} (i) & \dot{p} = c - ap - b(1 + \frac{\alpha_1}{\alpha_2})x_1, \quad p(0) = p_0 \\ (ii) & \dot{\xi} = (r+a)\xi - (1 + \frac{\alpha_1}{\alpha_2})x_1 \\ (iii) & p - \alpha_1 x_1 - b\xi = 0 \\ & \text{and} \\ (iv) & \xi(T) = 0 \quad \text{for a finite horizon problem} \\ & \text{or} \\ (iv)' & \lim_{t \rightarrow \infty} e^{-rt} \xi(t) = 0 \quad \text{for an infinite horizon problem.} \end{array} \right.$$

These conditions can also be expressed in terms of x_2 by making use of (6.6).

Following a procedure similar to (4.9)-(4.13), it can be shown that for a finite horizon problem, ξ and p are related by the linear time varying expression

$$\xi(t) = M(t)p(t) + N(t)$$

where $M(t)$, and $N(t)$ are the solution of

$$(6.8) \left\{ \begin{array}{l} \dot{M} = M(r + 2a + b(\frac{1}{\alpha_1} + \frac{1}{\alpha_2})(1 - bM)) - (\frac{1}{\alpha_1} + \frac{1}{\alpha_2})(1 - bM) \\ \dot{N} = (r + a + b(\frac{1}{\alpha_1} + \frac{1}{\alpha_2})(1 - bM))N - Mc \\ \text{with } M(T) = 0 \quad \text{and} \quad N(T) = 0. \end{array} \right.$$

The collusive supply curves in closed-loop form are:

$$(6.9) \quad x_i^*(t, p(t)) = \frac{1}{\alpha_i} (1-bM)p(t) - \frac{b}{\alpha_i} N, \quad i = 1, 2.$$

If the time horizon is infinite, then the relationship between ξ and p become time invariant:

$$(6.10) \quad \xi(t) = \bar{M} p(t) + \bar{N}$$

where $\bar{M}$ and $\bar{N}$ are constants satisfying the algebraic equations

$$(6.11) \quad \begin{cases} M(r+2a+b(\frac{1}{\alpha_1} + \frac{1}{\alpha_2})(1-bM)) - (\frac{1}{\alpha_1} + \frac{1}{\alpha_2})(1-bM) = 0 \\ (r+a+b(\frac{1}{\alpha_1} + \frac{1}{\alpha_2})(1-bM))\bar{N} - Mc = 0. \end{cases}$$

The collusive supply curves in this case are

$$(6.12) \quad x_i^*(p(t)) = \frac{1}{\alpha_i} (1-b\bar{M})p(t) - \frac{b}{\alpha_i} \bar{N}, \quad i = 1, 2$$

and in order to show that the boundary conditions (6.4) (v)' are satisfied, we note that by simple algebraic manipulations it can be shown that (6.11) has a solution $\bar{M}, \bar{N}$ satisfying

$$(6.13) \quad 0 \leq \bar{M} \leq \frac{1}{b}, \quad \bar{N} \geq 0$$

and hence, the collusive price trajectory which satisfies,

$$(6.14) \quad \dot{p} = (c + b^2(\frac{1}{\alpha_1} + \frac{1}{\alpha_2})\bar{N}) - (a + b(\frac{1}{\alpha_1} + \frac{1}{\alpha_2})(1-b\bar{M}))p(t)$$

approaches asymptotically the equilibrium price

$$(6.15) \quad \bar{p} = \frac{c + b^2 \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) \bar{N}}{a + b \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) (1 - b\bar{M})} > 0.$$

Thus

$$\lim_{t \rightarrow \infty} e^{-rt} \xi_i(t) = \lim_{t \rightarrow \infty} e^{-rt} [\bar{M}_i p^*(t) + \bar{N}_i] = 0.$$

The existence of the equilibrium collusive price is then always guaranteed.

Let us now obtain the infinite time horizon collusive solution for the numerical example which was considered in Section 5. With the given parameters the solution of (6.11) is

$$\bar{M} = 0.30 \quad \text{and} \quad \bar{N} = 0.312.$$

The collusive supply curves (6.12) are:

$$(6.16) \quad \begin{cases} x_1^*(p(t)) = -0.467 + 1.05 p(t) \\ x_2^*(p(t)) = -0.312 + 0.70 p(t) \end{cases}$$

and the price trajectory is

$$(6.17) \quad p^*(t) = (p_0 - 1.27)e^{-3.75t} + 1.27$$

which has an equilibrium level $\bar{p} = 1.27$.

For the purpose of comparison the equilibrium levels of the results obtained from the Cournot and the collusion solutions for the numerical example are summarized in Table 1. The first three columns in this table refer to the equilibrium levels of the price and the outputs of both firms. The last three columns refer to the equilibrium values of the undiscounted profits

			Equilibrium Price	Equilibrium Output $\bar{x}_1$	Equilibrium Output $\bar{x}_2$	Equilibrium Profit/t for Firm 1	Equilibrium Profit/t for Firm 2	Equilibrium Joint Profit
Cournot	Dynamic	Closed- Loop	1.06	1.07	0.81	0.75	0.53	1.28
		Open- Loop	1.13	0.98	0.75	0.79	0.57	1.36
	Static		1.14	0.97	0.76	0.78	0.58	1.36
Collusion	Dynamic	Closed-loop = Open-loop	1.27	0.86	0.58	0.84	0.57	1.41
	Static		1.29	0.86	0.57	0.86	0.57	1.43

Table 1

per unit time (i.e. $\bar{p} \bar{x}_1 - \frac{1}{2} \alpha_1 \bar{x}_1^2$). These would represent a measure of the future profits after the initial transient price adjustment period has died away. Two additional reference strategies, static Cournot and static collusion strategies, have also been added to the table. These were computed on the basis that price reacts instantaneously to changes in outputs (i.e. $\dot{p} = 0$ and $p = \frac{c}{a} - \frac{b}{a} (x_1 + x_2)$).

The following specific conclusions are imminent:

- (i) The closed-loop Cournot equilibrium is most desirable from the consumer welfare point of view. The open-loop Cournot equilibrium is less desirable than the closed-loop counterpart, and the collusive monopolist equilibrium is the least desirable. As far as 'desirability' is concerned, the firms may see it in the reverse order.
- (ii) The static Cournot solution is very close to or almost identical to the open-loop Cournot at equilibrium, and the static collusive monopolist exhibits almost the same type of price and output strategy as its dynamic counterpart. This implies that whether the adjustment is instantaneous or dynamic, as long as they ignore additional information in their decision-making, they end up, after the initial transient period, with (almost) the same equilibrium strategy.

In this paper, we have not analyzed the effect of change of the discount rate on the dynamic Cournot paths and the Cournot equilibrium. However, we are studying this aspect and a new result is expected shortly.

Conclusion

It seems that the biggest hurdle in the way of formulating the dynamic duopoly game was the intrinsic difficulty in establishing the dynamic relationship between the market supply and demand functions. In this paper the concept of a dynamic demand function was introduced as a means of reflecting the time delays (or dynamics) associated with the reaction in the price whenever the market supply is changed. This dynamic relationship is certainly more natural than the well established static one where price reacts instantaneously to a change in supply. The profit functions over a certain time horizon are then readily defined and the resulting dynamic duopoly game extends naturally as a problem within the framework of differential game theory. The necessary conditions for the solution of this problem are then developed for general dynamic demand functions and cost functions. These conditions yield some general results, but not specific enough to give us quantitative information. The linear-demand quadratic-cost functions case was then studied in detail and a numerical example is solved for the closed-loop and open-loop Cournot paths and equilibria. In order to make our analysis complete the dynamic collusive solution is then studied, and formulated as an optimal control problem. The collusive solution is then obtained for the same numerical example and some welfare implications of the equilibrium solutions are then discussed.

There are several other aspects of our dynamic duopoly game that are still subject to further investigation. These are

- i) To evaluate (analytically if possible) the effect of the discount rate on the Cournot and collusion paths and equilibria.
- ii) To build a model that can take care of inventory accumulation and decumulation.
- iii) To explore the possibility of formulating combined closed-loop, open-loop strategies in order to handle cases where price measurements are obtained at discrete instants of time (for example at the end of each month) rather than continuously (see Simaan and Cruz (1973c)).
- iv) To study and develop other strategies such as the Stackelberg strategy etc..... in a dynamic framework.

Finally, as a natural extension of these efforts, our model may serve as a starting point for stochastic analysis of dynamic duopoly games. When the system dynamics contain structural stochastic components or when the parameters of the system are subject to some stochastic law, we may be able to solve, for instance by following Aoki (1973), for stochastic Cournot or collusion controls. This way, the effect of uncertainties on the duopolists strategies, the equilibrium prices and the consumer welfare can be studied and some light can be shed on this vast and mostly unexplored field.

Appendix 1

From (3.2) the equation for λ_i can be written as

$$(A1) \quad \dot{\lambda}_i(t) = \left(r - \frac{\partial f}{\partial p} - \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial p} \right) \lambda_i(t) - x_i(t, p(t)) \quad , \quad \lambda_i(T) = 0$$

where $i=1, j=2$ or $i=2, j=1$. Under the assumption that $\frac{\partial x_j}{\partial p} \geq 0$, and (2.5) it follows that

$$(A2) \quad \left(r - \frac{\partial f}{\partial p} - \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial p} \right) \geq 0.$$

We will show that $\lambda_i(t) \geq 0$ by contradiction. Assume that for some t , $\lambda_i(t) < 0$, then since $\lambda_i(T) = 0$, there must exist a t_1 such that $\lambda_i(t_1) < 0$ and $\dot{\lambda}_i(t_1) > 0$. But this is a contradiction since such a $\lambda_i(t)$ will never satisfy A_1 . Identical proofs hold for $\gamma_i(t)$ and $\xi_i(t)$.

Appendix 2

Proof of Proposition 1: Let $\mathcal{K}$ be a compact subset of the real line $\mathbb{R}$ defined by

$$\mathcal{K} = \{K \in \mathbb{R} : 0 \leq K \leq \frac{1}{b}\}.$$

Now rearrange Eq. (4.15 (a) and (a)') in the following form

$$(B1) \quad \begin{cases} (a) & K_1^2 - \Psi_1(K_2)K_1 + \frac{1}{b^2} = 0 \\ (a)' & K_2^2 - \Psi_2(K_1)K_2 + \frac{1}{b^2} = 0 \end{cases}$$

where

$$(B2) \quad \begin{cases} (a) & \Psi_1(K_2) = \frac{\alpha_1}{b^2} \left[r + 2a + \frac{2b}{\alpha_1} + \frac{2b}{\alpha_2} (1 - bK_2) \right] \\ (a)' & \Psi_2(K_1) = \frac{\alpha_2}{b^2} \left[r + 2a + \frac{2b}{\alpha_2} + \frac{2b}{\alpha_1} (1 - bK_1) \right]. \end{cases}$$

Now for each $K_2 \in \mathcal{K}$, a solution to (B1 (a)) is

$$(B3) \quad K_1 = \frac{1}{2} \Psi_1(K_2) \pm \sqrt{\frac{1}{4} \Psi_1^2(K_2) - \frac{1}{b^2}}$$

and the following inequalities can be shown by simple algebraic manipulations on (B2(a)):

$$(B4) \quad \left\{ \begin{array}{l} (i) \quad \Psi_1(K_2) \geq 0 \\ (ii) \quad \frac{1}{4} \Psi_1^2(K_2) - \frac{1}{2} \geq 0 \\ (iii) \quad \frac{1}{2} \Psi_1(K_2) \geq \sqrt{\frac{1}{4} \Psi_1^2(K_2) - \frac{1}{2}} \geq \frac{1}{2} \Psi_1(K_2) - \frac{1}{b} . \end{array} \right.$$

Thus from (B4(iii)) it follows that K_1 in (B3) satisfies

$$0 \leq K_1 \leq \frac{1}{b}$$

or $K_1 \in \mathcal{K}$. Therefore (B3) defines a continuous mapping from $\mathcal{K} \rightarrow \mathcal{K}$.

Similarly for each $K_1 \in \mathcal{K}$, a solution to (B1(a)') is

$$(B5) \quad K_2 = \frac{1}{2} \Psi_2(K_1) - \sqrt{\frac{1}{4} \Psi_2^2(K_1) - \frac{1}{2}}$$

and by the same argument, (B5) defines a continuous mapping from $\mathcal{K} \rightarrow \mathcal{K}$.

Thus by the Brower fixed point theorem, there exists a pair $(\bar{K}_1, \bar{K}_2)$, $\bar{K}_1 \in \mathcal{K}$ and $\bar{K}_2 \in \mathcal{K}$ which satisfies (B3) and (B5) or (B1(a)) and (B1(a)') simultaneously.

Since $\bar{p}_c > 0$, (see footnote 4), it follows from (4.16) that $\bar{p}_c > 0$ and this along with (4.15b - b') implies that $E_1 \geq 0$ and $E_2 \geq 0$.

The proof of Proposition 2, is almost identical to the above and therefore it will be omitted.

References

- M. Aoki, "On Some Price Adjustment Schemes of a Marketeer," a paper presented at the NBER Conference on Stochastic Control and Economic Systems, U. of Chicago, June 7-9, 1973.
- K. Arrow, "Applications of Control Theory to Economic Growth," The Stanford Institute for Mathematical Studies in the Social Sciences, Stanford University, 1968.
- S. Clemhout, G. Leitman and H. Y. Wan, Jr., "A Differential Game Model of Duopoly," Econometrica, 39, Nov. 1971, 911-938.
- A. Cournot, Researches into the Mathematical Principles of the Theory of Wealth, New York, 1897 (original):
- R. M. Cyert and M. H. DeGroot, (1973a), "An Analysis of Cooperation and Learning in a Duopoly Context," Amer. Econ. Rev., March 1973, 24-37.
- _____ and _____, (1973b), "Sequential Strategies in Dual Control Problem," a paper presented at the NBER Conference on Stochastic Control and Economic Systems, U. of Chicago, June 7-9, 1973.
- J. W. Friedman, "Reaction Function and the Theory of Duopoly," Rev. Econ. Stud., July 1968, 35, 257-272.
- Y. C. Ho, "Differential Games, Dynamic Optimization and Generalized Control Theory," Jour. Opt. Theo. and Appl., Sept. 1970, 6, 179-209.
- M. D. Intriligator, Mathematical Optimization and Economic Theory, Englewood Cliffs, 1971.
- H. W. Kuhn and G. P. Szegö, Differential Games and Related Topics, Amsterdam, 1971.
- H. Nikaido, Introduction to Sets and Mappings in Modern Economics, Amsterdam, 1970.
- D. K. Osborne, "The Duopoly Game: Output Variations," Amer. Econ. Rev., Sept. 1971, 41, 538-560.
- L. S. Pontryagin, et al.; The Mathematical Theory of Optimal Processes, New York, 1962.
- P. A. Samuelson, Foundation of Economic Analysis, Cambridge, 1947.
- M. Shubik, Strategy and Market Structure, New York, 1959.

- M. Simaan and J. B. Cruz, Jr., (1973a), "On the Stackelberg Strategy in Nonzero-Sum Games," Jour. Opt. Theo. and Appl., May 1973, 11, 533-555.
- _____ and _____, (1973b), "Additional Properties of the Stackelberg Strategy in Nonzero-Sum Games," Jour. Opt. Theo. and Appl., June 1973, 11, 613-626.
- _____ and _____, (1973c), "Sampled Data Nash Controls in Nonzero-Sum Differential Games," Int. Jour. Cont., June 1973, 17, 1201-1209.
- J. Simon, C. Puig, and J. Aschoff, "A Duopoly Simulation and Richer Theory: An End to Cournot " Rev. Econ. Stud., July 1973, 40, 353-366.
- A. W. Starr and Y. C. Ho, (1969a), "Nonzero-Sum Differential Games," Jour. Opt. Theo. and Appl., March 1969, 3, 184-206.
- _____ and _____, (1969b), "Further Properties of Nonzero-Sum Differential Games," Jour. Opt. Theo. and Appl., April 1969, 3, 207-219.
- J. von Neuman and D. Morgenstern, Theory of Games and Economic Behavior, Princeton, 1953.
- H. von Stackelberg, Marktform und Gleichgewicht, Vienna, 1934.

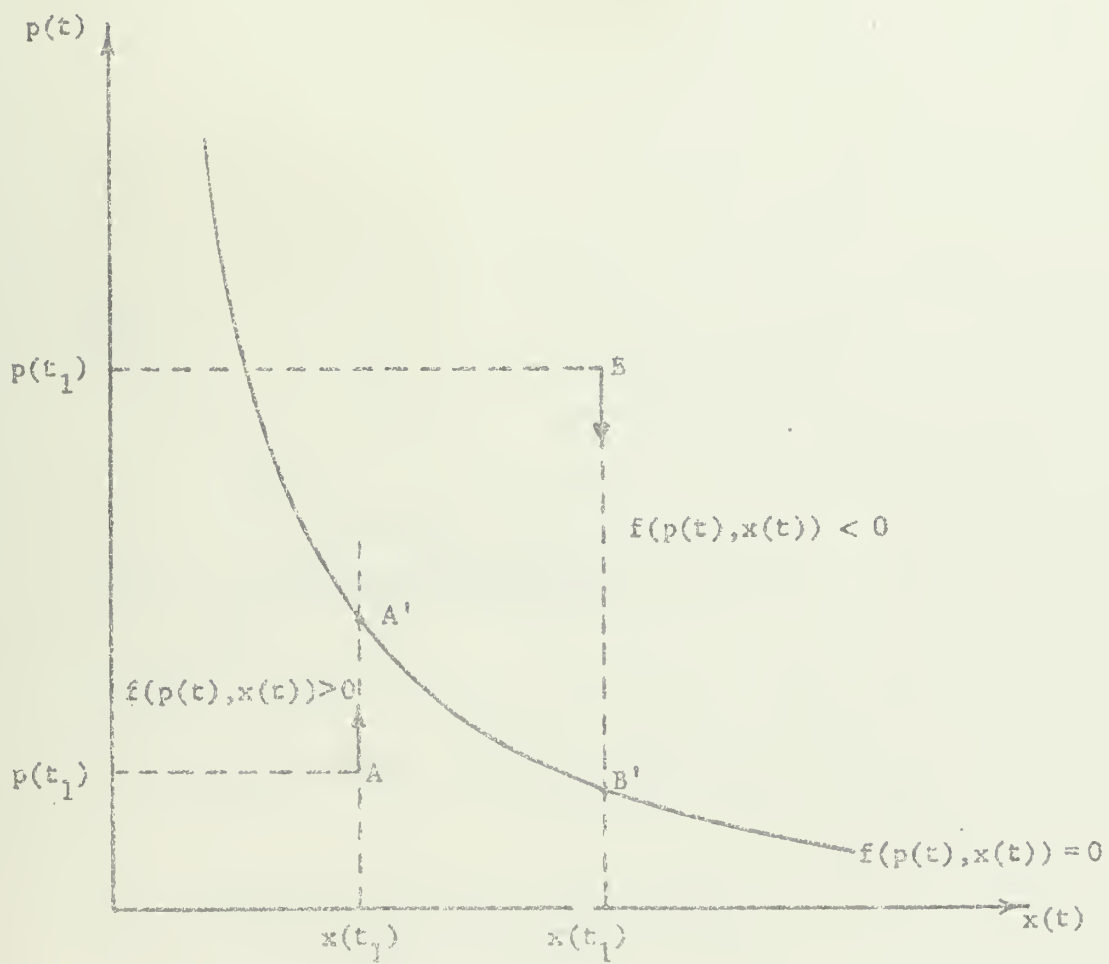


Fig. 1. Dynamic Demand Function

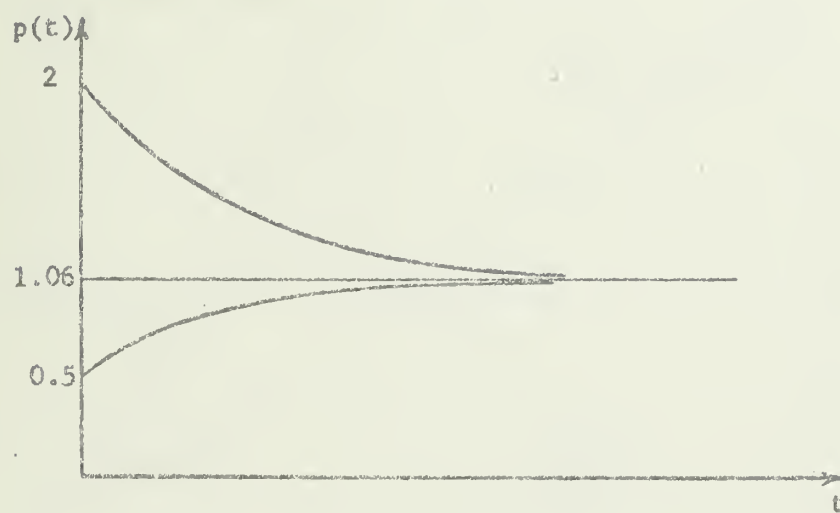


Fig. 2. Time path of $p(t)$ when $t \in [0, \infty)$

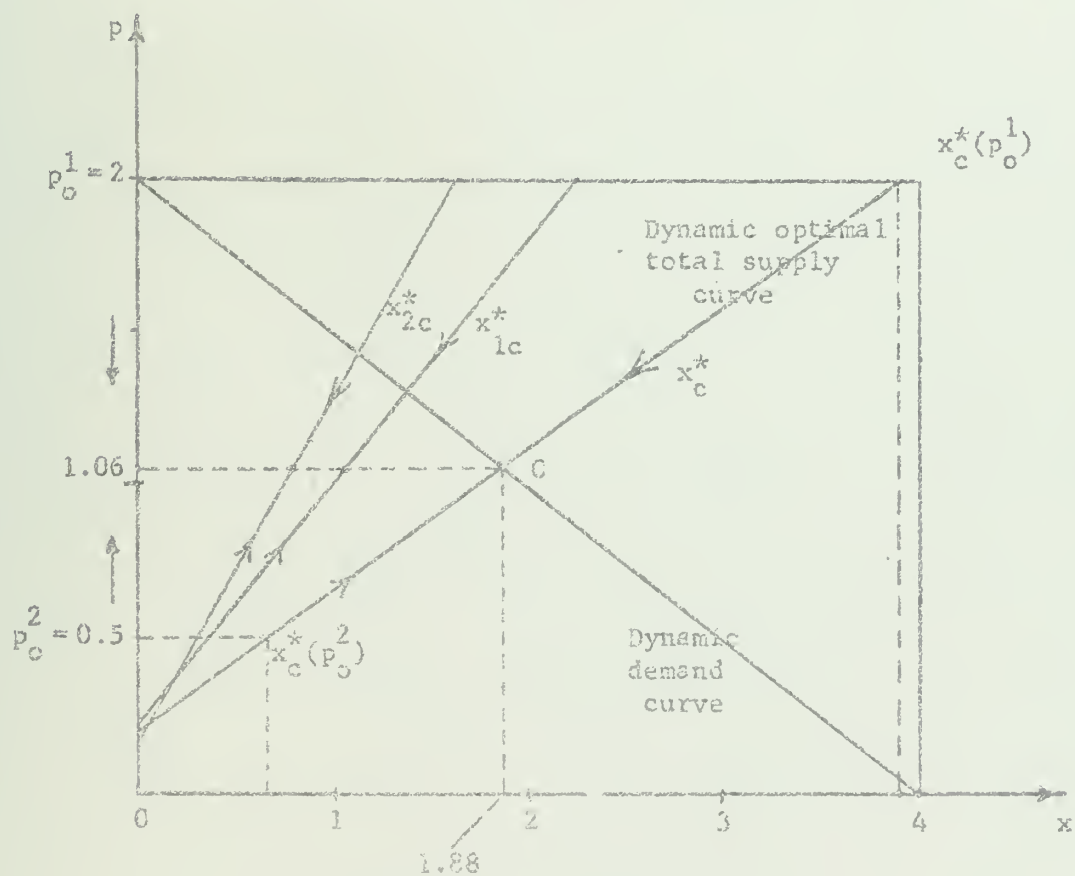


Fig. 3. Closed-loop optimal supply curves, $x_c^*(p(t)) = x_{1c}^*(p(t)) + x_{2c}^*(p(t))$, and the closed-loop Cournot Equilibrium point C.



UNIVERSITY OF ILLINOIS-URBANA



3 0112 060296800